Operad.

Operad is build to understand when a space has the homotopy type of a loop space or more generally n-fold loop space. Informally, an operad $\mathcal{O}$ is a collection of different arities with rules of how to compose them. An algebra over $\mathcal{O}$ gives a space X together with operations on X induced by those of $\mathcal{O}$.

Given a space X with a multiplication $\mu: X \times X \to X$ which is not associative. Then $\mu(\mu(-,-),-)$ and $\mu(-,\mu(-,-))$ are different, We consider the space $\mathcal{O}(3)$ of ternary operations, then the geometry of $\mathcal{O}(3)$ tells us how badly the associativity fails. In fact, if μ is strictly associative, then $\mathcal{O}(3)$ is just a point. The next best thing is to require $\mathcal{O}(3)$ to be contractible. This inspires the best kind of associativity right after strict associativity: all $\mathcal{O}(n)$'s are contractible, then we say the multiplication is A_∞.

non-Σ operad

Def: A non-Σ operad is a collection of spaces $\{\mathcal{O}(n)\}_{n\geq 0}$ with

1) there are maps
$$\circ_{n,m_1,\dots,m_n} : \mathcal{O}(n) \otimes \mathcal{O}(m_1) \otimes \dots \otimes \mathcal{O}(m_n) \to \mathcal{O}(\sum m_i)$$
(composition law)

2) there is a special element in $\mathcal{O}(1)$ corresponding to identity.

3) $\mathcal{O}(0) = *$

And there are some standard associtivity and unital compatibility conditions.

Let X be a based space, then $\mathcal{O}$ acts on X if there is a morphism of operads $\mathcal{O} \to End_X$, where End_X is the endomorphism operad consists of maps $X^n \to X$.

<u>Ex.</u> The associative operad Ass, defined by $Ass(n) = *$ $\forall n$. Then an action of Ass on a space X is a map ~~$\theta(n)$~~ $Ass(n) \to End_X(n)$ $\forall n$, which picks out a single operation $X^n \to X$ $\forall n$. Hence X is an Ass-algebra $\Leftrightarrow$ X has a unital, associative multiplication.

<u>Ex.</u> $X = \Omega Y$ contatenation of loops gives a multiplication which is not associative due to parametrization. But there are homotopy between different parametrization.

e.g. $a(bc) =$ 

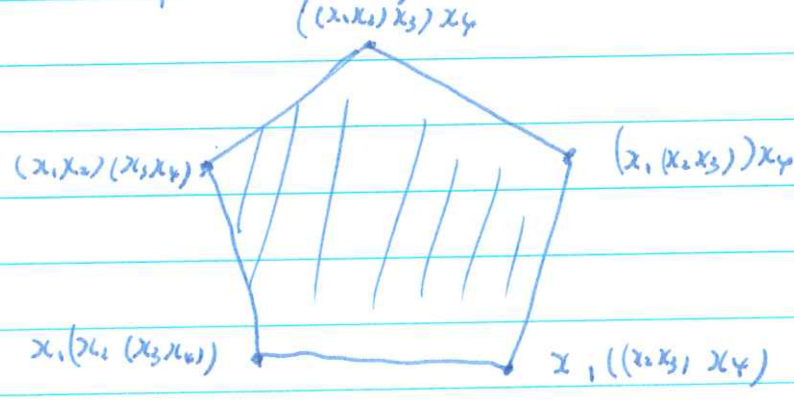
$\simeq (ab)c =$ 

similarly, there are five ways to multiply four loops, with not only homotopies between pairs, but also higher homotopies.

In fact, this is an algebra over a non-Σ operad K, called the Stasheff associahedron. Here are first several components. $K(1) = K(2) = *$,

$K(3) =$ 

$K(4)$ is the fill-in pentagon labelled.



In particular, $K(n)$ is contractible.
Hence, we have the map $K(3) \to End_X(3)$.
As a consequence, all Ass-algebra's are K-algebras.

We call X an A_∞-space if it has an action of an A_∞ [non-Σ] operad, i.e an operad $\mathcal{O}$ such that all $\mathcal{O}(n)$ are contractible.
In fact, X is A_∞ iff it has an action of K.

<u>THM</u> (Recognition principle, $n=1$ case) every group like A_∞ space has the weak homotopy type of a loop space.

Next, we want to look at commutativity.

<u>Def:</u> An operad $\mathcal{O}$ is a non-Σ operad with an action of symmetric group Σ_n on each $\mathcal{O}(n)$, and with some additional coherence relations.

E.g. The commutative operad Comm, $Comm(n) = *$

check a comm-algebra is equivalent an unital, associative, and commutative ~~algebra~~ monoid.

Def: An E_∞ operad is an operad with $E(n)$ contractible $\forall n$, and Σ_n action is free.

E.g. $E\Sigma_n$ is contractible w/ a free Σ_n-action, then we can define $\mathcal{E}(n) = E\Sigma_n$ with structure maps coming from maps of Σ_i's. $\mathcal{E}$ is called the Barratt-Eccles operad.

Given a non-Σ operad, we can turn it into an operad C^Σ by $C^\Sigma(n) = C(n) \times \Sigma_n$ with Σ_n acts freely on the right.

Ex. $K^\Sigma(2)$ has two points x_1x_2 and x_2x_1

$K^\Sigma(3)$ has 6 copies of intervals.

$(x_1x_2)x_3$ — $x_1(x_2x_3)$ $\quad$ $(x_1x_3)x_2$ — $x_1(x_3x_2)$ $\quad$ ......

An and En. Recall ~~we know a "not associative multiplication"~~ ~~which~~ the definition of homotopy associative multiplication (i.e. $\exists$ homotopy between $\mu(\mu(-,-),-)$ and $\mu(-,\mu(-,-))$). This is a condition on $K(3)$. In fact.

homotopy associative $\Leftrightarrow$ $\exists$ action of $K(i)$, $i \leq 3$.

i.e. we have a truncated operad K_3.

Given an H-space, we can write down coherent maps from $\{K(i)\}_{i \leq 2}$ when $K(2)$ picks out a multiplication and ~~[illegible]~~ structure map make it unital w.r.t. base point. Then we can talk about A_n-space.

We get a hierarchy

$$\begin{array}{ccccccccc} X \text{ is an } H \text{ space} & \Leftarrow & X \text{ has homotopy asso. mult.} & \Leftarrow & \cdots & \Leftarrow & X \text{ is } A_n & \Leftarrow \cdots \Leftarrow & X \text{ is } A_\infty \\ & & & & & & & & \Uparrow \\ & & & & & & & & X \text{ is strictly associative} \end{array}$$

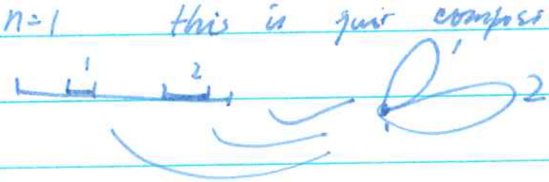
We want to do the same thing for E_∞.

Def: (little n-cube operad) $\mathcal{C}_n$ is defined by

$\mathcal{C}_n(j) : \underbrace{I^n \sqcup I^n \sqcup \cdots I^n}_{j \text{ copies}} \longrightarrow I^n$ which specify j-little cubes in I^n that do not overlap.

Note: $\mathcal{C}_n$ acts on $\Omega^n X$. Note $\Omega^n X$ is the space of maps $(I^n, \partial I^n) \to (X, *)$, for j little cubes $c_1, \ldots, c_j$ and j maps $f_i : (I^n, \partial^n) \to (\partial I^n)$, we can define a map f

e.g. $n=1$ this is just composition of j loops.



lemma: $\mathcal{C}_1$ is an A_∞ operad.

Then we have the Hierarchy

$$A_\infty = E_1 \Leftarrow E_2 \Leftarrow \cdots \Leftarrow E_\infty \Leftarrow \text{strictly commutative}$$

Note: $E_2 \Leftrightarrow$ homotopy commutative.

Monad associated to $\mathcal{C}$.

Given an operad $\mathcal{C}$, we can define a monad (C, μ, η) where C is an endofunctor $Top_* \to Top_*$

$$CX = \coprod_{j\geq 0} \mathcal{C}(j)_+ \wedge_{\Sigma_j} X^{\wedge j}$$

explicitly, $CX = \sqcup_{j\geq 0} \mathcal{C}(j) \times X^j / \sim$

in the case $\mathcal{C} = End_X$, $\sim$ means

1) "inclusion of a base point works well":

$$\left(f : X^n \to X \right) \sim \left(\tilde{f} : X^{n-1} \xrightarrow{x_j = *} X^n \right)$$

$$(a_1, \cdots, \underset{\substack{\uparrow \\ j\text{-th}}}{*}, \cdots, a_{n-1}) \qquad (a_1, \cdots, a_{n-1})$$

2) permuting the variables in f is the same as permuting the order of copies of X in X^j.

<u>Rmk</u>: This monad arises from the free-forgetful adjunction $Top_* \rightleftarrows \mathcal{C}$-algebras. Moreover, the category of C algebras is equivalent to the category of $\mathcal{C}$ algebras.

Ex. (James reduced product).
Consider Ass^{Σ}, where $Ass^{\Sigma}(j) = Ass(j) \times \Sigma_j \simeq \Sigma_j$.
We have $AX = \sqcup_j Ass^{\Sigma}(j) \times_{\Sigma_j} \cancel{X}^j / \sim$. Every element (σ, x) is equivalent to a unique representative of the form $(1, y)$. Hence we can regard elements of AX as order strings $x_1, \cdots, x_n$, and the base point condition guarentees the compatibility with inclusion of the base point anywhere in the string. Here.

$$AX \cong \bigvee_j X^{\wedge j}$$

Consider the little cubes operad $\mathcal{C}_n$, which acts on $\Omega^n X$. Let (C_n, μ_n, η_n) be the corresponding ~~of~~ monad it induces a map $\theta_n : C_n \Omega^n X \to \Theta_n \Omega^n X$.

THM. $\exists$ a map of monads $C_n \to \Omega^n \Sigma^n$ by
$$\alpha_n : C_n \xrightarrow{C_n \eta_n} C_n \Omega^n \Sigma^n X \xrightarrow{\theta_n} \Omega^n \Sigma^n X$$
which is a weak equivalence.

THM (Recognition principal) $\forall\, 1 \le n \le \infty$, group-like E_n-space. (i.e. C_n-algebras) are the same as n-fold loop spaces.

Instead of ~~top~~ spaces, we can consider operad of other categories, e.g. spectra, simplicial sets, chain complexes, etc.

Recall, a spectrum in connective if it has homotopy groups in only nonnegative dimensions. Hence we have an equivalence between connective spectra and grouplike E_∞-space.

Suppose $\mathcal{O}$ is an operad, An $\mathcal{O}$-ring spectrum R is one which equipped with maps.
$$\mathcal{O}(k)_+ \wedge R_{n_1} \wedge \cdots \wedge R_{n_k} \to R_{\sum_{i=1}^{k} n_i}$$
with appropriate coherence relations.
So we can speak of A_∞-ring spectra and E_∞-ring spectra.
In general, it's difficult to construct such ring structures on a given spectrum.

We have following examples of E_∞-ring spectra

1. Suspension spectra of E_∞-spaces.
2. Thom spectra of infinite loop maps e.g. MO, MU etc.
3. KO, KU, algebraic K-theory spectrum of a comm. ring.